

Painlevé Test and Higher Order Differential Equations

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Abstract

Starting from the second Painlevé equation, we obtain Painlevé type equations of higher order by using the singular point analysis.

1 Introduction

Painlevé and his school [20, 21, 14, 15] investigated second-order first-degree equations of the form

$$y'' = F(z, y, y'), \quad (1.1)$$

where F is rational in y' , algebraic in y and locally analytic in z , and has no movable critical points. This property is known as the Painlevé property and ordinary differential equations (ODE), which possess it, are said to be of Painlevé type. They found that, within a Möbius transformation, there exist fifty such equations. Distinguished among these fifty equations are the six Painlevé equations, PI, ..., PVI. Any other of the fifty equations either can be integrated in terms of known functions or can be reduced to one of these six equations.

Higher order first-degree in the polynomial class and higher order higher degree equations of Painlevé type were investigated by Fuchs [13, 15], Briot and Bouquet [15], Chazy [5], Bureau [3, 4], Exton [11], Martynov [17], Cosgrove [8, 9], Kudryashov [16], Clarkson, Joshi and Pickering [6] and also in the articles [22, 23, 19, 18]. The Riccati equation is

the only example for the first-order first-degree equation which has the Painlevé property. The best known third order equation is Chazy's natural-barrier equation

$$y''' = 2yy'' - 3y'^2 + \frac{4}{36-n^2} (6y' - y^2)^2, \quad (1.2)$$

The case $n = \infty$ appears in several physical problems. The equation (1.2) is integrable for all real and complex n and $n = \infty$. Its solutions are rational for real n in $2 \leq n \leq 5$, and have a circular natural barrier for $n \geq 7$ and $n = \infty$.

In this article the second Painlevé hierarchy is investigated by using the Painlevé ODE test or singular point analysis. It is possible to obtain the equation of Painlevé type of any order, as well as the known ones, starting from the second Painlevé equation. Painlevé ODE test which is an algorithm introduced by Ablowitz, Ramani, Segur (ARS) [1, 2] tests whether a given ODE satisfies the necessary conditions to be of Painlevé type.

The procedure to obtain higher order Painlevé type equations starting from the second Painlevé equation may be summarized as follows:

I. Take an n^{th} order Painlevé type differential equation

$$y^{(n)} = F(z, y, y', \dots, y^{(n-1)}), \quad (1.3)$$

where F is analytic in z and rational in its other arguments. If $y \sim y_0(z - z_0)^\alpha$ as $z \rightarrow z_0$, then α is a negative integer for certain values of y_0 . Moreover, the highest derivative term is one of the dominant terms. Then the dominant terms are of order $\alpha - n$. There are n resonances $r_0 = -1, r_1, r_2, \dots, r_{n-1}$, with all r_i , $i = 1, 2, \dots, (n-1)$ being nonnegative real distinct integers such that $Q(r_j) = 0$, $j = 0, 1, 2, \dots, (n-1)$. The compatibility conditions for the simplified equation that retains only dominant terms of (1.3) are identically satisfied. Differentiation of the simplified equation with respect to z yields

$$y^{(n+1)} = G(z, y, y', \dots, y^{(n)}), \quad (1.4)$$

where G contains the terms of order $\alpha - n - 1$, and the resonances of (1.4) are the roots of $Q(r_j)(\alpha + r - n) = 0$. Hence equation (1.4) has a resonance $r_n = n - \alpha$ additional to the resonances of (1.3). Equation (1.4) passes the Painlevé test provided that $r_n \neq r_i$, $i = 1, 2, \dots, (n-1)$ and is a positive integer. Moreover the compatibility conditions are identically satisfied, that is $z_0, y_{r_1}, \dots, y_{r_n}$ are arbitrary.

II. Add the dominant terms which are not contained in G . Then the resonances of the new equation are the zeros of a polynomial $\tilde{Q}(r)$ of order $n+1$. Find the coefficients of $\tilde{Q}(r)$ such that there is at least one principal Painlevé branch, that is, all $n+1$ resonances (except $r_0 = -1$) are real positive distinct integers for at least one possible choice of (α, y_0) . The other possible choices of (α, y_0) may give the secondary Painlevé branch, that is, all the resonances are distinct integers.

III. Add the nondominant terms which are the terms of weight less than $\alpha - n - 1$, with coefficients analytic in z . Find the coefficients of the nondominant terms by the use of the compatibility conditions and transformations which preserve the Painlevé property.

In this article we consider only the principal branch, that is, all the resonances r_i (except $r_0 = -1$) are positive real distinct integers and the number of resonances is equal to the

order of the differential equation for a possible choice of (α, y_0) . Then, the compatibility conditions give full set of arbitrary integration constants. The other possible choices of (α, y_0) may give secondary branches which possess several distinct negative integer resonances. Negative but distinct integer resonances give no conditions which contradict integrability [12]. Higher order equations with negative but distinct integer resonances might be investigated separately. In the present work we start with the second Painlevé equation and obtain the third, fourth and some fifth order equations of Painlevé type. All of the third order and some of the fourth order equations are found in the literature, but for the sake of completeness we present the known equations with appropriate references. A similar procedure was used in [18] to obtain the higher order equations of Painlevé type by starting from the first Painlevé equation. The procedure can also be used to obtain the higher order equations by starting from PIII, \dots , PVI. These results will be published elsewhere.

2 Third order equations: $P_{II}^{(3)}$

The second Painlevé equation, PII, is

$$y'' = 2y^3 + zy + \nu. \quad (2.1)$$

The Painlevé test gives that there is only one branch and the resonances are $(-1, 4)$. The dominant terms of (2.1) are y'' and $2y^3$ which are of order -3 as $z \rightarrow z_0$. Differentiation of the simplified equation $y'' = 2y^3$ gives

$$y''' = ay^2y', \quad (2.2)$$

where a is a constant which can be introduced by replacing y with λy , such that $6\lambda^2 = a$. Addition of the polynomial type terms of order -4 gives the following simplified equation

$$y''' = a_1yy'' + a_2y'^2 + a_3y^2y' + a_4y^4, \quad (2.3)$$

where a_i , $i = 1, \dots, 4$ are constants. Substitution of

$$y = y_0(z - z_0)^{-1} + \beta(z - z_0)^{r-1}, \quad (2.4)$$

into the simplified equation (2.3) gives the following equations, $Q(r) = 0$, for the resonance r and for y_0 , respectively,

$$Q(r) = (r+1) \{ r^2 - (a_1y_0 + 7)r - [a_3y_0^2 - 2(2a_1 + a_2)y_0 - 18] \} = 0, \quad (2.5a)$$

$$a_4y_0^3 - a_3y_0^2 + (2a_1 + a_2)y_0 + 6 = 0. \quad (2.5b)$$

Equation (2.5b) implies that, in general, there are three branches of Painlevé expansions if $a_4 \neq 0$. Now one should determine y_{0j} , $j = 1, 2, 3$, and a_i such that at least one of the branches is the principal branch. There are three cases which should be considered separately.

Case I. $a_3 = a_4 = 0$: In this case y_0 takes a single value, i.e. there is only one branch. Equation (2.5a) implies that $r_0 = -1$ and $r_1 r_2 = 6$. Therefore there are the following four possible cases:

- a. $y_{01} = -6/a_2 : (r_1, r_2) = (1, 6), \quad a_1 = 0,$
 - b. $y_{01} = -2/a_1 : (r_1, r_2) = (2, 3), \quad a_1 = a_2,$
 - c. $y_{01} = -12/a_1 : (r_1, r_2) = (-2, -3), \quad a_1 = -2a_2/3,$
 - d. $y_{01} = -14/a_1 : (r_1, r_2) = (-1, -6).$
- (2.6)

The case d is not be considered since $r = -1$ is a double resonance. The compatibility conditions are identically satisfied for the first two cases. To find the canonical form of the third-order equations of Painlevé type, one should add nondominant terms the coefficients of which are analytic functions of z , that is, one should consider the following equation for each case

$$y''' = a_1 y y'' + a_2 y'^2 + A_1 y'' + A_2 y y' + A_3 y^3 + A_4 y' + A_5 y^2 + A_6 y + A_7, \quad (2.7)$$

where $A_k(z)$, $k = 1, \dots, 7$, are analytic functions of z . Substitution of

$$y = y_0(z - z_0)^{-1} + \sum_{j=1}^6 y_j(z - z_0)^{j-1}, \quad (2.8)$$

into equation (2.7) gives the recursion relation for y_j . Then one can find A_k such that the recursion relation, i.e. the compatibility conditions for $j = r_1, r_2$, are identically satisfied and hence y_{r_1}, y_{r_2} are arbitrary.

I.a: The Painlevé property is preserved under the following linear transformation

$$y(z) = \mu(z)u(t) + \nu(z), \quad t = \rho(z), \quad (2.9)$$

where μ, ν and ρ are analytic functions of z . By using the transformation (2.9) one can set $A_4 = A_5$, $A_1 = 0$, and $a_2 = -6$. The compatibility condition at the resonance $r_1 = 1$ gives $A_2 = A_3$. The arbitrariness of y_1 in the recursion relation for $j = 6$ and the recursion relation yield that

$$A_5'' - A_5^2 = 0, \quad A_6'' - A_5 A_6 = 0, \quad A_7'' - \frac{1}{3} A_5 A_7 = \frac{1}{6} A_6^2, \quad A_3 = 0. \quad (2.10)$$

According to the equation (2.10a) there are three cases should be considered separately.

I.a.i: $A_5 = 0$: From equation (2.10) all coefficients A_k can be determined uniquely. The canonical form of the third order equation for this case is

$$y''' = -6y'^2 + (c_1 z + c_2)y + \frac{1}{72}c_1^2 z^4 + \frac{1}{18}c_1 c_2 z^3 + \frac{1}{12}c_2^2 z^2 + c_3 z + c_4, \quad (2.11)$$

where c_i , $i = 1, \dots, 4$ are constants.

If $c_1 = c_2 = 0$, then (2.11) can be written as

$$u'' = 6u^2 - c_3 z - c_4, \quad (2.12)$$

where $u = -y'$. If $c_3 = 0$, then the solution of (2.12) can be written in terms of elliptic functions. If $c_3 \neq 0$, (2.12) can be transformed into the first Painlevé equation.

If $c_1 = 0$ and $c_2 \neq 0$, (2.11) takes the following form by replacing y by γy and z by δz such that $\gamma\delta = 1$ and $c_2\delta^3 = 6$

$$y''' = -6y'^2 + 6y + 3z^2 + \tilde{c}_3 z + \tilde{c}_4, \quad (2.13)$$

where $\tilde{c}_3 = c_3\delta^5$, $\tilde{c}_4 = c_4\delta^4$. Equation (2.13) was given in [5] and [3].

If $c_1 \neq 0$ and $c_2 = 0$, replacement of y by γy and z by δz in (2.11) such that $\gamma\delta = 1$ and $c_1\delta^4 = 12$ yields

$$y''' = -6y'^2 + 12zy + 2z^4 + \tilde{c}_3 z + \tilde{c}_4, \quad (2.14)$$

where $\tilde{c}_3 = c_3\delta^5$, $\tilde{c}_4 = c_4\delta^4$. Equation (2.14) was given by Chazy [5] and Bureau [3]. It should be noted that (2.11) can be reduced to (2.14) by the replacement of z by $z - (c_2/c_1)$ and then replacing y by γy and z by δz such that $\gamma\delta = 1$, $c_1\delta^4 = 12$.

I.a.ii: $A_5 = \frac{6}{(z+c)^2}$: Without loss of generality the integration constant c can be set to zero.

From (2.10) the coefficients A_k can be determined and the canonical form of the equation is

$$y''' = -6y'^2 + 6z^{-2}(y' + y^2) + (c_1 z^3 + c_2 z^{-2})y + c_3 z^2 + c_4 z^{-1} + \frac{1}{18} \left(\frac{1}{18} c_1^2 z^8 + \frac{3}{2} c_1 c_2 z^3 + \frac{3}{4} c_2^2 z^{-2} \right), \quad (2.15)$$

where c_i , $i = 1, \dots, 4$, are constants.

If $c_1 = c_2 = 0$, (2.15) is a special case of the equation given by Chazy [5]. If $c_1 = 0$ and $c_2 \neq 0$, (2.15) takes the following form by replacing y by γy and z by δz such that $\gamma\delta = 1$ and $c_2\delta = 24$

$$y''' = -6y'^2 + 6z^{-2}(y' + y^2 + 4y) + \tilde{c}_3 z^2 + \tilde{c}_4 z^{-1} + 24z^{-2}, \quad (2.16)$$

where $\tilde{c}_3 = c_3\delta^6$ and $\tilde{c}_4 = c_4\delta^3$. Equation (2.16) is given in [3].

If $c_1 \neq 0$ and $c_2 = 0$, then equation (2.15) takes the form

$$y''' = -6y'^2 + \frac{6}{z^2}(y' + y^2) + 18z^3 y + z^8 + \tilde{c}_3 z^2 + \tilde{c}_4 \frac{1}{z}, \quad (2.17)$$

where \tilde{c}_3, \tilde{c}_4 are constants. Equation (2.17) was given in [3].

I.a.iii: If one replaces A_5 with $6\hat{A}_5$, A_6 with $6\hat{A}_6$ and A_7 with $6\hat{A}_7$, then equations (2.10) yields

$$\hat{A}_5'' - 6\hat{A}_5^2 = 0, \quad \hat{A}_6'' - 6\hat{A}_5\hat{A}_6 = 0, \quad \hat{A}_7'' - 2\hat{A}_5\hat{A}_7 = \hat{A}_6^2. \quad (2.18)$$

Integration of (2.18a) once gives

$$(\hat{A}_5')^2 = 4\hat{A}_5^3 - \alpha_1, \quad (2.19)$$

where α_1 is an integration constant. Then

$$\hat{A}_5 = \mathcal{P}(z; 0, \alpha_1), \quad (2.20)$$

where \mathcal{P} is the Weierstrass elliptic function. If $\hat{A}_6 = 0$, (2.18c) implies that \hat{A}_7 satisfies Lamé's equation. Hence

$$\hat{A}_7 = c_1 E_1(z) + c_2 F_1(z), \quad (2.21)$$

where c_1 and c_2 are constants, $E_1(z)$ and $F_1(z)$ are the Lamé functions of degree one of the first and second kind, respectively. They are given as

$$E_1(z) = e^{-z\zeta(a)} \frac{\sigma(z+a)}{\sigma(z)}, \quad F_1(z) = e^{z\zeta(a)} \frac{\sigma(z-a)}{\sigma(z)}, \quad (2.22)$$

where ζ is the ζ -Weierstrass function such that $\zeta' = -\mathcal{P}(z)$, σ is the σ -Weierstrass function such that $\frac{\sigma'(z)}{\sigma(z)} = \zeta(z)$ and a is a parameter such that $\mathcal{P}(a; 0, \alpha_1) = 0$. Then the equation

$$y''' = -6y'^2 + 6\mathcal{P}(z; 0, \alpha_1)(y' + y^2) + \tilde{c}_1 e^{-z\zeta(a)} \frac{\sigma(z+a)}{\sigma(z)} + \tilde{c}_2 e^{z\zeta(a)} \frac{\sigma(z-a)}{\sigma(z)}, \quad (2.23)$$

where $\tilde{c}_1 = 6c_1$ and $\tilde{c}_2 = 6c_2$. Equation (2.23) was considered in [5]. If $\hat{A}_6 \neq 0$, $\mathcal{P}(z; 0, \alpha_1)$ also solves equation (2.18b). Then (2.18c) implies that \hat{A}_7 satisfies the nonhomogeneous Lamé's equation. Hence,

$$\hat{A}_7(z) = k_1(z)E_1(z) + k_2(z)F_1(z), \quad (2.24)$$

where

$$k_1(z) = k_1 - \int_z \frac{\mathcal{P}^2(t; 0, \alpha_1)}{W(t)} F_1(t) dt, \quad k_2(z) = k_2 + \int_z \frac{\mathcal{P}^2(t; 0, \alpha_1)}{W(t)} E_1(t) dt \quad (2.25)$$

with k_1 and k_2 are constants of integration and $W(z) = E_1 F_1' - E_1' F_1$.

I.b: The coefficients $A_k(z)$, $k = 1, \dots, 7$, of the nondominant terms can be found by using the linear transformation (2.9) and the compatibility conditions. The linear transformation (2.9) allows one to set $a_2 = -2$, $A_1(z) = 0$, $A_2(z) = A_3(z)$ and the compatibility conditions give that $A_2(z) = A_6(z) = 0$ and $A_4(z) = A_5(z)$. So the canonical form of the equation is

$$y''' = -2(y y'' + y'^2) + A_4(y' + y^2) + A_7, \quad (2.26)$$

where A_4 and A_7 are arbitrary analytic functions of z . If one lets $u = y' + y^2$, then (2.26) can be reduced to a linear equation for u . Equation (2.26) was given in [3].

I.c: Without loss of generality one can choose $a_1 = 2$. Then the simplified equation is

$$y''' = 2y y'' - 3y'^2, \quad (2.27)$$

which was also considered in [5, 3].

Case II. $a_4 = 0$: In this case y_0 satisfies the quadratic equation

$$a_3 y_0^2 - (2a_1 + a_2) y_0 - 6 = 0. \quad (2.28)$$

Therefore there are two branches corresponding to $(-1, y_{0j})$, $j = 1, 2$. The resonances satisfy equation (2.5a). Now one should determine y_{0j} and a_i , $i = 1, 2, 3$ such that one of the branches is the principal branch.

If y_{0j} are the roots of (2.28), by setting

$$P(y_{0j}) = -[a_3 y_{0j}^2 - 2(2a_1 + a_2)y_{0j} - 18], \quad j = 1, 2, \quad (2.29)$$

and if (r_{j1}, r_{j2}) are the resonances corresponding to y_{0j} , then one has

$$r_{j1}r_{j2} = P(y_{0j}) = p_j, \quad j = 1, 2, \quad (2.30)$$

where p_j are integers such that at least one is positive. Equation (2.28) gives that

$$a_3 = -\frac{6}{y_{01}y_{02}}, \quad 2a_1 + a_2 = a_3(y_{01} + y_{02}). \quad (2.31)$$

Then (2.29) can be written as

$$P(y_{01}) = 6 \left(1 - \frac{y_{01}}{y_{02}}\right), \quad P(y_{02}) = 6 \left(1 - \frac{y_{02}}{y_{01}}\right). \quad (2.32)$$

For $p_1 p_2 \neq 0$, the p_j satisfy the following Diophantine equation

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{6}. \quad (2.33)$$

Now one should determine all finite integer solutions of Diophantine equation. One solution of (2.33) is $(p_1, p_2) = (12, 12)$. The following cases should be considered: **i)** If $p_1 > 0$, $p_2 > 0$ and $p_1 < p_2$, then $p_1 > 6$ and $p_2 > 12$. **ii)** If $p_1 > 0$, $p_2 < 0$, then $p_1 < 6$. Based on these observations there are the following nine integer solutions of Diophantine equation, viz

$$(p_1, p_2) = (12, 12), (7, 42), (8, 24), (9, 18), (10, 15), \\ (2, -3), (3, -6), (4, -12), (5, -30). \quad (2.34)$$

For each (p_1, p_2) , one should write (r_{j1}, r_{j2}) such that r_{ji} are distinct integers and $r_{j1}r_{j2} = p_j$, $j = 1, 2$. Then y_{0j} and a_i can be obtained from (2.31) and (2.32) and

$$r_{j1} + r_{j2} = a_1 y_{0j} + 7, \quad j = 1, 2, \quad (2.35)$$

There are the five following cases such that all the resonances are distinct integers for both branches. The resonances and the simplified equations for these cases are

$$\text{II.a: } a_1 = a_2 = 0, \quad y_{01}^2 = \frac{6}{a_3} : (r_{11}, r_{12}) = (3, 4), \quad y_{02} = -y_{01} : (r_{21}, r_{22}) = (3, 4), \\ y''' = a_3 y^2 y', \quad (2.36)$$

$$\text{II.b: } y_{01} = -\frac{1}{a_1} : (r_{11}, r_{12}) = (2, 4), \quad y_{02} = \frac{3}{a_1} : (r_{21}, r_{22}) = (4, 6), \\ y''' = a_1 (y y'' + 2y'^2 + 2a_1 y^2 y'), \quad (2.37)$$

$$\begin{aligned} \text{II.c: } y_{01} &= -\frac{3}{a_1} : (r_{11}, r_{12}) = (1, 3), \quad y_{02} = -\frac{6}{a_1} : (r_{21}, r_{22}) = (-2, 3), \\ y''' &= a_1 \left(yy'' + y'^2 - \frac{1}{3} a_1 y^2 y' \right), \end{aligned} \quad (2.38)$$

$$\begin{aligned} \text{II.d: } y_{01} &= -\frac{2}{a_1} : (r_{11}, r_{12}) = (1, 4), \quad y_{02} = -\frac{6}{a_1} : (r_{21}, r_{22}) = (-3, 4), \\ y''' &= a_1 \left(yy'' + 2y'^2 - \frac{1}{2} a_1 y^2 y' \right), \end{aligned} \quad (2.39)$$

$$\begin{aligned} \text{II.e: } y_{01} &= -\frac{1}{a_1} : (r_{11}, r_{12}) = (1, 5), \quad y_{02} = -\frac{6}{a_1} : (r_{21}, r_{22}) = (-5, 6), \\ y''' &= a_1 (yy'' + 5y'^2 - a_1 y^2 y'). \end{aligned} \quad (2.40)$$

For each case the compatibility conditions for the simplified equations are identically satisfied. To find the canonical form of the third order equations of Painlevé type, one should add nondominant terms with the coefficients which are analytic functions of z .

II.a: By using the linear transformation (2.9) one can set $2A_1 + A_3 = 0$, $A_2 = 0$ and $a_3 = 6$. The compatibility conditions at $j = 3, 4$, for the both branches allow one to determine the coefficients A_k . The canonical form of the equation for this case is

$$\begin{aligned} y''' &= 6y^2 y' - \left(\frac{1}{2} c_1^2 z^2 - c_2 z - c_3 \right) y' + c_1 y^2 \\ &\quad - (c_1^2 z - c_2) y - \frac{1}{4} c_1^3 z^2 + \frac{1}{2} c_1 c_2 z + \frac{1}{2} c_1 c_3, \end{aligned} \quad (2.41)$$

where c_1, c_2, c_3 are constants. If one replaces $z - \frac{c_3}{c_1^2}$ by z , y by γy and z by δz such that $\gamma\delta = 1$ and $c_1\delta^2 = -2$, then (2.41) yields

$$u''' = 6u^2 u' + 12zuu' + 4(z^2 + k)u' + 4zu + 4u^2, \quad (2.42)$$

where $u = y - z$ and k is a constant. Equation (2.42) was considered in [5, 3], and its first integral is PIV.

If $c_1 = c_2 = 0$, (2.41) can be solved in terms of elliptic functions. If $c_1 = 0$ and $c_2 \neq 0$, (2.41) gives

$$y''' = 6y^2 y' + c_2 \left(z + \frac{c_3}{c_2} \right) y' + c_2 y. \quad (2.43)$$

If one introduces $t = z + \frac{c_3}{c_2}$, then the first integral of (2.43) is PII.

II.b: By using the linear transformation (2.9) one can always choose $2A_1 + A_3 = 0$, $A_2 = 0$ and $a_1 = -1$. Then the compatibility conditions for the both branches, that is the arbitrariness of y_{21} and y_{41} for the first branch and y_{42} and y_{62} for the second branch, imply that all the coefficients A_k of nondominant terms, are zero. So the canonical form for this case is

$$y''' = -yy'' - 2y'^2 + 2y^2 y'. \quad (2.44)$$

Equation (2.44) was given in [5, 3].

II.c: By using the linear transformation (2.9) one can always set $A_3 = A_5 = 0$ and $a_1 = -3$. Then the compatibility conditions at $j = 1, 3$ give that $A_1 = c_1/2$, $A_2 = c_1$, $c_1 = \text{constant}$ and $A'_4 = A_6$. Then the canonical form of the equation is

$$y''' = -3yy'' - 3y'^2 - 3y^2y' + \frac{1}{2}c_1y'' + c_1yy' + A_4y' + A'_4y + A_7. \quad (2.45)$$

The first integral of (2.45) gives that

$$u'' = -3uu' - u^3 + B_1u + B_2, \quad (2.46)$$

where $u = y - (c_1/6)$, and B_1 and B_2 are arbitrary analytic functions of z . Equation (2.45) was considered in [5].

II.d: One can always choose $A_3 = A_5 = 0$ and $a_1 = -2$ by the linear transformation (2.9). The arbitrariness of y_{11} and y_{41} for the first branch and y_{42} for the second branch imply that $A_1 = A_2 = A_7 = 0$ and $A'_4 = 2A_6$. The canonical form is

$$y''' = -2yy'' - 4y'^2 - 2y^2y' + A_4y' + \frac{1}{2}A'_4y. \quad (2.47)$$

The first integral of (2.47) is

$$y'' = \frac{y'^2}{2y} - 2yy' - \frac{y^3}{2} + A_4y + \frac{c}{y}, \quad c = \text{constant}. \quad (2.48)$$

The equation (2.47) was considered in [5, 3].

II.e: By the linear transformation (2.9) one can choose $A_1 = A_3 = 0$ and $a_1 = -1$. Then the compatibility conditions give that $A_2 = A_5 = 0$, $A_6 = A'_4/3$ and $A_7 = -A''_4/3$. After the replacement of y by $-y$ and A_4 by $3A_4$ the canonical form of the equation for this case is

$$y''' = yy'' + 5y'^2 - y^2y' + 3A_4y' + A'_4y + A''_4. \quad (2.49)$$

Equation (2.49) has the first integral

$$\begin{aligned} (y'' - yy' - y^3 + A_4y + A'_4)^2 &= \frac{8}{3}(y' - y^2) \left(y' + \frac{y^2}{2} + \frac{3}{2}A_4 \right) \\ &+ 4(y' - y^2)(2A_4y^3 + A'_4y + A''_4) + 4A_4^2y^2 + 4A_4A'_4y + 4A_4'^2 + c, \end{aligned} \quad (2.50)$$

where A_4 is an arbitrary function of z and c is an arbitrary constant of integration. Equation (2.49) was also considered in [5, 3].

Case III. $a_4 \neq 0$: In this case there are three branches corresponding to $(-1, y_{0j})$, $j = 1, 2, 3$, where y_{0j} are the roots of (2.5b). Equation (2.5b) implies that

$$\sum_{j=1}^4 y_{0j} = \frac{a_3}{a_4}, \quad \sum_{i \neq j} y_{0i}y_{0j} = \frac{1}{a_4}(2a_1 + a_2), \quad \prod_{j=1}^3 y_{0j} = -\frac{6}{a_4}. \quad (2.51)$$

If the resonances (except $r_0 = -1$, which is common for all branches) are r_{ji} , $i = 1, 2$, corresponding to y_{0j} and if one sets

$$P(y_{0j}) = -[a_3y_{0j}^2 - 2(2a_1 + a_2)y_{0j} - 18], \quad j = 1, 2, 3, \quad (2.52)$$

then (2.5a) implies that

$$\prod_{i=1}^2 r_{ji} = P(y_{0j}) = p_j, \quad (2.53)$$

where p_j are integers and in order to have a principal branch at least one of them is positive. Equations (2.51) and (2.52) give

$$\begin{aligned} p_1 &= 6 \left(1 - \frac{y_{01}}{y_{02}}\right) \left(1 - \frac{y_{01}}{y_{03}}\right), & p_2 &= 6 \left(1 - \frac{y_{02}}{y_{01}}\right) \left(1 - \frac{y_{02}}{y_{03}}\right), \\ p_3 &= 6 \left(1 - \frac{y_{03}}{y_{01}}\right) \left(1 - \frac{y_{03}}{y_{02}}\right) \end{aligned} \quad (2.54)$$

and hence, the p_j satisfy the following Diophantine equation

$$\sum_{j=1}^3 \frac{1}{p_j} = \frac{1}{6}. \quad (2.55)$$

Moreover equation (2.54) gives that

$$\prod_{j=1}^3 p_j = -\frac{6^3}{(y_{01}y_{02}y_{03})^2} (y_{01} - y_{02})^2 (y_{01} - y_{03})^2 (y_{02} - y_{03})^2, \quad (2.56)$$

if $a_1 \neq 0$, that is, if $p_1 > 0$, then either p_2 or p_3 is a negative integer. One should consider the case $a_1 = 0$ separately.

III.a: $a_1 = 0$: In this case the sum of the resonances for all three branches is fixed and

$$\sum_{i=1}^2 r_{ji} = 7, \quad j = 1, 2, 3. \quad (2.57)$$

Under this condition the solutions of the Diophantine equation (2.55) are $(p_1, p_2, p_3) = (10, 10, -30)$ and $(10, 12, -60)$.

III.a.i: $(p_1, p_2, p_3) = (10, 10, -30)$: Equation (2.54) can be written as

$$p_1(y_{02} - y_{03}) = ky_{01}, \quad p_2(y_{03} - y_{01}) = ky_{02}, \quad p_3(y_{01} - y_{02}) = ky_{03}, \quad (2.58)$$

where

$$k = \frac{6}{y_{01}y_{02}y_{03}} (y_{01} - y_{02})(y_{02} - y_{03})(y_{01} - y_{03}). \quad (2.59)$$

For $k = \pm 10\sqrt{5}$ the system (2.58) has the nontrivial solutions y_{0j} , $j = 1, 2, 3$. For these values of y_{0j} the resonances and the coefficients a_i , $i = 2, 3, 4$, are

$$\begin{aligned} y_{01} &= \nu \left(1 - \sqrt{5}\right) : (r_{11}, r_{12}) = (2, 5), & y_{02} &= \nu \left(1 + \sqrt{5}\right) : (r_{21}, r_{22}) = (2, 5), \\ y_{03} &= 6\nu : (r_{31}, r_{32}) = (-3, 10), \\ a_2 &= \frac{2}{\nu}, & a_3 &= \frac{2}{\nu^2}, & a_4 &= \frac{1}{4\nu^3}, & \nu &= \text{constant}, \end{aligned} \quad (2.60)$$

for both values of k . For these values of y_{0j} and a_i the simplified equation passes the Painlevé test for all branches. The linear transformation (2.9) and the compatibility conditions at the resonances of the first and second branches are enough to determine all coefficients $A_k(z)$ of the nondominant terms. The canonical form of the equation for this case is,

$$y''' = 12y'^2 + 72y^2y' + 54y^4 + c_1, \quad (2.61)$$

where c_1 is an arbitrary constant. Equation (2.61) can be obtained with the choice of $\nu = 1/(1 - \sqrt{5})$ and replacement of y with $6y/(1 - \sqrt{5})$. Equation (2.61) was given in [5, 8].

III.a.ii: $(p_1, p_2, p_3) = (10, 12, -60)$: For this case equation (2.58) has nontrivial solution y_{0j} for $k = \pm 20\sqrt{3}$. Then y_{0j} , a_i and the corresponding resonances are as follows:

$$\begin{aligned} y_{01} &= -\frac{1}{\nu} \left(-1 \pm \sqrt{3} \right) : (r_{11}, r_{12}) = (2, 5), \quad y_{02} = \pm \frac{\sqrt{3}}{\nu} : (r_{21}, r_{22}) = (3, 4), \\ y_{03} &= -\frac{1}{\nu} \left(-6 \pm \sqrt{3} \right) : (r_{31}, r_{32}) = (-5, 12), \\ a_2 &= 3 \frac{7 \pm 3\sqrt{3}}{11} \nu, \quad a_3 = \frac{40 \pm 14\sqrt{3}}{11} \nu^2, \quad a_4 = \frac{7 \pm 3\sqrt{3}}{11} \nu^3, \quad \nu = \text{constant}. \end{aligned} \quad (2.62)$$

By using the linear transformation (2.9) one can choose $\nu = \pm\sqrt{3}$ and $A_1 = A_2 = 0$. All other coefficients A_k of the nondominant terms can be determined from the compatibility conditions at the resonances of the first and second branches. The canonical form for this case is as follows:

$$\begin{aligned} y''' &= \frac{27 \pm 21\sqrt{3}}{11} (y'^2 + y^4) + \frac{120 \pm 42\sqrt{3}}{11} y^2 y' \\ &\quad + c \left(\pm \frac{1 \mp \sqrt{3}}{\sqrt{3}} y' + y^2 \right) - \frac{231 \pm 143\sqrt{3}}{132} c^2 \end{aligned} \quad (2.63)$$

or

$$\begin{aligned} y''' &= 6y^2 y' + \frac{3}{11} \left(9 \pm 7\sqrt{3} \right) (y' + y^2)^2 - \frac{1}{22} \left(4 \mp 3\sqrt{3} \right) c_1 y' \\ &\quad + \frac{1}{44} \left(3 \mp 5\sqrt{3} \right) c_1 y^2 - \frac{1}{352} \left(9 \pm 7\sqrt{3} \right) c_1^2, \end{aligned} \quad (2.64)$$

where $c_1 = 44c/(3 \mp 5\sqrt{3})$. Equation (2.64) was considered in [8].

III.b: $a_1 \neq 0$: Since $p_1, p_2 > 0$, $p_3 < 0$, equation (2.56) can be written as

$$p_1 p_2 \hat{p}_3 = 6n^2, \quad (2.65)$$

where n is a constant and $\hat{p}_3 = -p_3$. Then the Diophantine equation (2.55) yields

$$p_1 p_2 = \hat{p}_3 (p_1 + p_2) - n^2 \quad (2.66)$$

and, since $(p_1 - p_2)^2 \geq 0$, then

$$(p_1 + p_2)^2 - 4\hat{p}_3 (p_1 + p_2) + 4n^2 \geq 0. \quad (2.67)$$

Therefore $0 < \hat{p}_3 \leq n$. Hence one may assume that n is a positive integer. When $\hat{p}_3 = n$, equations (2.65) and (2.66) give $(p_1, p_2, p_3) = (6, n, -n)$ as the solution of the Diophantine equation. For the case of $\hat{p}_3 < n$, if one assumes that $p_1 < p_2$ (if $p_1 = p_2$, (2.66) implies that p_1 and p_2 are complex numbers), then the Diophantine equation (2.55) implies that $p_1 < 12$. Equations (2.55) and (2.66) give that

$$(p_1 \hat{p}_3)^2 = n^2 [6p_1 - (6 - p_1) \hat{p}_3], \quad (p_1 p_2)^2 = n^2 [6p_1 + (6 - p_1) p_2] \quad (2.68)$$

for $p_1 < 6$ and for $6 < p_1 < 12$ respectively. Equation (2.68) imply that $[6p_1 - (6 - p_1) \hat{p}_3]$ and $[6p_1 + (6 - p_1) p_2]$ must be squares of integers. By the use of these results, p_j the multiplication of the resonances for the branches corresponding the y_{0j} , $j = 1, 2, 3$, are

$$\begin{aligned} (p_1, p_2, p_3) = & (4, 6, -10), (5, 870, -26), (5, 195, -21), (7, 41, -1722), \\ & (7, 38, -399), (7, 33, -154), (8, 22, -264), (8, 16, -48), \\ & (9, 15, -90), (10, 14, -210), (11, 13, -858). \end{aligned} \quad (2.69)$$

For each values of (p_1, p_2, p_3) given in (2.69) one should follow the given steps below for $(4, 6, -10)$.

When $(p_1, p_2, p_3) = (4, 6, -10)$, $p_1 = 4$ implies that possible integral values of r_{1i} , $i = 1, 2$ are $(r_{11}, r_{12}) = (1, 4), (-1, -4)$. Then

$$r_{j1} + r_{j2} = a_1 y_{0j} + 7, \quad j = 1, 2, 3, \quad (2.70)$$

implies that $y_{01} = -2/a_1$ and $y_{01} = -12/a_1$ for $(r_{11}, r_{12}) = (1, 4), (-1, -4)$ respectively. On the other hand y_{0j} satisfies equation (2.58) for $k = \pm 20$. For $k = 20$, $y_{02} = -9y_{01}/14$, but the resonance equation for the second branch

$$r_{2i}^2 - (7 + a_1 y_{02}) r_{2i} + p_2 = 0, \quad i = 1, 2, \quad (2.71)$$

implies that $7 + a_1 y_{02}$ be an integer. So, in order to have integer resonances (r_{21}, r_{22}) for the second branch, $a_1 y_{02}$ has to be integral. A similar argument holds for the third branch, but for $k = 20$, both y_{01} and y_{02} are not integers. Also for $k = -20$ the resonances for the second and third branches are not integers. Following the same steps one cannot find the integral resonances for the second and third branches for all other cases of (p_1, p_2, p_3) given in (2.69).

When $(p_1, p_2, p_3) = (6, n, -n)$, equation (2.58) has a nontrivial solution y_{0j} for $k = \pm n$. We have $y_{01} = 0$, $y_{02} = y_{03}$ for $k = n$ and $y_{01} = 12\nu$, $y_{02} = \nu(6 - n)$, $y_{03} = \nu(6 + n)$ for $k = -n$, where ν is an arbitrary constant. Since $y_{01} = 0$ for $k = n$, this case is not be considered. For $k = -n$, $2a_1 + a_2$, a_3 and a_4 can be determined from equation (2.51) to be

$$2a_1 + a_2 = -\frac{180 - n^2}{2\nu(36 - n^2)}, \quad a_3 = -\frac{12}{\nu^2(36 - n^2)}, \quad a_4 = -\frac{1}{2\nu^3(36 - n^2)}. \quad (2.72)$$

Since $p_1 = 6$, then all possible distinct integral resonances for the first branch are $(r_{11}, r_{12}) = (-1, -6), (-2, -3), (1, 6), (2, 3)$. Because of the double resonance, $r_0 = r_{11} = -1$, the case $(-1, -6)$ is not considered. When $(r_{11}, r_{12}) = (1, 6)$, equation (2.70) implies that $a_1 = 0$. This case was considered in case III.a. For the other possible resonances, $(-2, -3)$

and $(2, 3)$, one can obtain the a_i , $i = 1, 2, 3, 4$, and y_{0j} , $j = 1, 2, 3$. Once the coefficients of the resonance equation (2.5a) are known one should look at the distinct integer resonances for the second and third branches. We have only two cases, such that all the resonances are distinct integers for all branches. The resonances and the corresponding simplified equations are as follows:

$$\begin{aligned} \text{III.b.i: } y_{01} &= -\frac{12}{a_1} : (r_{11}, r_{12}) = (-2, -3), \\ y_{02} &= -\frac{1}{a_1}(6 - n) : (r_{21}, r_{22}) = (1, n), \\ y_{03} &= -\frac{1}{a_1}(6 + n) : (r_{31}, r_{32}) = (1, -n), \\ y''' &= a_1 \left[yy'' + \frac{3(12 + n^2)}{2(36 - n^2)} y'^2 - \frac{12}{36 - n^2} a_1 y^2 y' + \frac{1}{2(36 - n^2)} a_1^2 y^4 \right], \quad n \neq 1, 6. \end{aligned} \quad (2.73)$$

It should be noted that as $n \rightarrow \infty$ the simplified equation reduces to (2.27).

$$\begin{aligned} \text{III.b.ii: } y_{01} &= -\frac{2}{a_1} : (r_{11}, r_{12}) = (2, 3), \\ y_{02} &= -\frac{1}{a_1} \left(1 - \frac{n}{6} \right) : (r_{21}, r_{22}) = (6, n/6), \\ y_{03} &= -\frac{1}{a_1} \left(1 + \frac{n}{6} \right) : (r_{31}, r_{32}) = (6, -n/6), \\ y''' &= a_1 \left[yy'' + \frac{468 - n^2}{36 - n^2} y'^2 - \frac{432}{36 - n^2} a_1 y^2 y' + \frac{108}{36 - n^2} a_1^2 y^4 \right], \quad n \neq 6, 36. \end{aligned} \quad (2.74)$$

The canonical form of the equations corresponding to the above cases can be obtained by adding the nondominant terms with the analytic coefficients A_k , $k = 1, \dots, 7$.

III.b.i: By using the transformation (2.9) one can set $A_3 = A_4 = 0$ and $a_1 = 2$. The compatibility conditions at the resonances imply that all the coefficients are zero except A_6 and A_7 which remain arbitrary for $n = 2$. For $n = 3$ A_7 is arbitrary and all the other coefficients are zero. For $n = 4, 5$ and 6 all coefficients A_k are zero. However, it was proved in [7] that for $n \geq 4$ the equation does not admit nondominant terms. The canonical forms of the equations for $n = 2$ and $n = 3$ are

$$y''' = 2yy'' + \frac{3}{2}y'^2 - \frac{3}{2}y^2y' + \frac{1}{8}y^4 + A_6y + A_7, \quad (2.75)$$

$$y''' = 2yy'' + \frac{7}{3}y'^2 - \frac{16}{9}y^2y' + \frac{4}{27}y^4 + A_7, \quad (2.76)$$

respectively. Equations (2.75) and (2.76) were given in [5] and [8], and both can be linearized by letting $y = -2u'/u$ and $y = -3u'/2u$ respectively.

III.b.ii: The linear transformation (2.9) and the compatibility conditions at the resonances of the first and second branches give the canonical form as

$$\begin{aligned} y''' &= -2yy'' + \frac{26 - 2m^2}{m^2 - 1}y'^2 + \frac{24}{m^2 - 1}(2y' + y^2)y^2 + A_5(y' + y^2) \\ &\quad - \frac{m^2 - 1}{48} \left(A_5'' - \frac{1}{2}A_5^2 \right) + c_1z + c_2, \end{aligned} \quad (2.77)$$

where $m = 6/n$, $m \neq 1, 6$, c_1 and c_2 are arbitrary constants and A_5 is an arbitrary function of z . Equation (2.77) was given in [5] and [8] and is equivalent to

$$y' + y^2 = \frac{m^2 - 1}{48} A_5 - \frac{m^2 - 1}{4} u, \quad u'' = 6u^2 + \frac{1}{4(m^2 - 1)} (c_1 z + c_2). \quad (2.78)$$

3 Fourth order equations: $P_{II}^{(4)}$

Differentiation of (2.3) with respect to z gives the terms $y^{(4)}$, yy''' , $y'y''$, y^2y'' , yy'^2 , y^3y' , all of which are of order -5 for $\alpha = -1$, as $z \rightarrow z_0$. Addition of the term y^5 , which is also of order -5 , gives the following simplified equation

$$y^{(4)} = a_1 yy''' + a_2 y'y'' + a_3 y^2 y'' + a_4 yy'^2 + a_5 y^3 y' + a_6 y^5, \quad (3.1)$$

where a_i , $i = 1, \dots, 6$, are constants. Substitution of (2.4) into (3.1) gives the following equations for resonance r and for y_0 , respectively,

$$Q(r) = (r+1) \{ r^3 - (11 + a_1 y_0) r^2 - [a_3 y_0^2 - (7a_1 + a_2) y_0 - 46] r - a_5 y_0^3 + 2(2a_3 + a_4) y_0^2 - 6(3a_1 + a_2) y_0 - 96 \} = 0, \quad (3.2a)$$

$$a_6 y_0^4 - a_5 y_0^3 + (2a_3 + a_4) y_0^2 - 2(3a_1 + a_2) y_0 - 24 = 0. \quad (3.2b)$$

Equation (3.2b) implies that in general there are four branches of Painlevé expansion, if $a_6 \neq 0$, corresponding to the roots y_{0j} , $j = 1, 2, 3, 4$. Now one should determine y_{0j} and a_i such that at least one of the branches is the principal branch. Depending on the number of branches there are four cases. Each case should be considered separately.

Case I. $a_5 = a_6 = 0$, $2a_3 + a_4 = 0$: In this case there is only one branch which should be the principal branch. There are following two subcases which will be considered separately.

I.a: $a_1 = 0$: In this case the equation (3.2a) gives that the resonances (r_1, r_2, r_3) (additional to $r_0 = -1$) satisfy $\sum_{i=1}^3 r_i = 11$, $\prod_{i=1}^3 r_i = 24$. Under these conditions the only possible distinct positive integer resonances are $(r_1, r_2, r_3) = (1, 4, 6)$. Then (3.2) implies that $a_3 = 0$ and $y_0 = -12/a_2$. Therefore the simplified equation is

$$y^{(4)} = a_2 y' y''. \quad (3.3)$$

To obtain the canonical form of the equation, one should add the nondominant terms, viz y''' , yy'' , y'' , y'^2 , $y^2 y'$, yy' , y' , y^4 , y^3 , y^2 , y , 1 , that is terms of order greater than -5 as $z \rightarrow z_0$ with coefficients $A_k(z)$, $k = 1, \dots, 12$, which are analytic functions of z . The coefficients A_k can be determined by using the linear transformation (2.9) and the compatibility conditions at the resonances. One can choose $a_2 = -12$, $A_2 = 0$ and $2A_3 - A_6 + A_9 = 0$ by the linear transformation (2.9). The compatibility conditions, that is the arbitrariness of y_1, y_4, y_6 , give that

$$\begin{aligned} A_3'' - A_3^2 &= 0, & A_1' + A_1^2 &= A_3/3, & A_4 &= 6A_1, & A_5 &= A_8 = A_9 = 0, \\ A_6 &= 2A_3, & A_7'' - A_3 A_7 &= 2A_1 A_3 A_1' + 2A_1^2 A_3', & A_{10} &= A_3' - A_1 A_3, \\ A_{11} &= (A_7 - A_{10})' - A_1(A_7 - A_{10}), & A_{12}' + A_1 A_{12} &= \frac{1}{6}(A_7 - A_{10})^2. \end{aligned} \quad (3.4)$$

According to the solution of (3.4a), there are the four following cases:

I.a.i: $A_3 = 0$, $A_1 = 0$: Then the canonical form of the equation is

$$y^{(4)} = -12y'y'' + (c_1z + c_2)y' + c_1y + \frac{1}{18c_1}(c_1z + c_2)^3 + c_3, \quad (3.5)$$

where c_i , $i = 1, 2, 3$, are arbitrary constants. Integration of (3.5) once gives

$$y''' = -6y'^2 + (c_1z + c_2)y + \frac{1}{72c_1^2}(c_1z + c_2)^4 + c_3z + c_4, \quad (3.6)$$

where c_4 is an integration constant. If $c_1 \neq 0$ and $c_2 = 0$, then the equation (3.6) takes the form of (2.14). For $c_1 = 0$ and $c_2 \neq 0$ (3.6) yields (2.13).

I.a.ii: $A_3 = 0$, $A_1 = 1/(z - c)$: Without loss of generality one can choose the constant of integration c as zero. Then the canonical form of the equation is

$$\begin{aligned} y^{(4)} = & -12y'y'' + \frac{1}{z}y''' + \frac{6}{z}y'^2 + (c_1z - c_2)y' + \frac{c_2}{z}y + \frac{1}{24}c_1^2z^3 \\ & - \frac{1}{9}c_1c_2z^2 + \frac{1}{12}c_2^2z + \frac{c_3}{z}. \end{aligned} \quad (3.7)$$

If $c_1 = c_2 = 0$, then (3.7) is equivalent to

$$u' = \frac{1}{z}(u + c_3), \quad y''' = -6y'^2 + u. \quad (3.8)$$

If $c_1 = 0$ and $c_2 \neq 0$, after replacement of z by γz , y by βy , such that $\beta\gamma = 1$ and $c_2\gamma^3 = 6$ equation (3.7) takes the form

$$y^{(4)} = -12y'y'' + \frac{1}{z}y''' + \frac{6}{z}(y'^2 + y) - 6y' + 3z + \frac{\tilde{c}_3}{z}, \quad (3.9)$$

where $\tilde{c}_3 = c_3\gamma^4$. If $c_1 \neq 0$ and $c_2 = 0$, the equation (3.7) takes the form

$$y^{(4)} = -12y'y'' + \frac{1}{z}(y''' + 6y'^2) + 12zy' + 6z^3 + \frac{\tilde{c}_3}{z}, \quad (3.10)$$

where \tilde{c}_3 is an arbitrary constant.

I.a.iii: $A_3 = 6/(z - c)^2$: For simplicity let $c = 0$. Then the canonical form of the equation is

$$\begin{aligned} y^{(4)} = & -12y'y'' + A_1(y''' + 6y'^2) + \frac{6}{z^2}(y'' + 2yy') \\ & + A_7y' + A_{10}y^2 + A_{11}y + A_{12}, \end{aligned} \quad (3.11)$$

where

$$\begin{aligned} A_1 = & \frac{2c_1z^3 - c_2}{z(c_1z^3 + c_2)}, \\ A_7 = & \frac{1}{c_1z^3 + c_2} \left(\frac{1}{5}c_1c_3z^6 + \frac{1}{5}c_2c_3z^3 + c_1c_4z - 24c_1 + c_2c_4z^{-2} - 6c_2z^{-3} \right), \\ A_{10} = & -\frac{12}{z^3} - \frac{6}{c_1z^3 + c_2} (2c_1 - c_2z^{-3}), \end{aligned}$$

$$\begin{aligned}
A_{11} &= \frac{1}{c_1 z^3 + c_2} \left(\frac{c_1 c_3^2}{1350} z^{10} + \frac{c_2 c_3^2}{900} z^7 + \frac{c_1 c_3 c_4}{60} z^5 + \frac{c_2 c_3 c_5}{15} z^2 \right. \\
&\quad \left. + c_5 z - \frac{c_1 c_4^2}{6} - \frac{c_2 c_4^2}{24} z^{-3} \right), \\
A_{12} &= \frac{-1}{(c_1 z^4 + c_2 z)^2} \left[c_1^2 c_3 z^{10} - 48 c_1^2 z^8 + \frac{4 c_1 c_2 c_3}{5} z^7 + 5 c_1^2 c_4 z^5 - \frac{c_2^2 c_3}{5} z^4 \right. \\
&\quad \left. + 4 c_1 c_2 c_4 z^2 - 42 c_1 c_2 z - c_2^2 c_3 z^{-1} + 6 c_2^2 z^{-2} - (c_1 z^4 + c_2 z) \right. \\
&\quad \left. \times \left(\frac{6 c_1 c_3}{5} z^6 + \frac{3 c_2 c_3}{5} z^3 - 48 c_1 + c_1 c_4 - 2 c_2 c_4 z^{-2} + 6 c_2 z^{-3} \right) \right], \quad (3.12)
\end{aligned}$$

where c_i , $i = 1, \dots, 5$, are constants. Equations (3.5), (3.7), (3.9), (3.10) and (3.11) were considered in [3, 9, 17].

I.a.iv: $A_3 = 6\mathcal{P}(z; 0, \alpha_1)$: If one replaces A_3 with $6\hat{A}_3$ in the equation (3.4a), then Weierstrass elliptic function $\mathcal{P}(z; 0, \alpha_1)$, where α_1 is an arbitrary integration constant, is a solution of the resulting equation. If one lets $A_1 = \tilde{A}'_1/\tilde{A}_1$, then the equation (3.4b) gives Lamé's equation for \tilde{A}_1 . Therefore,

$$\tilde{A}_1 = c_1 E_1(z) + c_2 F_1(z), \quad (3.13)$$

where c_1 and c_2 are constants and $E_1(z)$ and $F_1(z)$ are the Lamé functions of degree one and of the first and second kind respectively, which were given in (2.22). Similarly replacement of A_1 with $6\hat{A}_1$, A_3 with $6\hat{A}_3$ and A_7 with $6\hat{A}_7$ in the equation for A_7 in (3.4) yields

$$\hat{A}_7'' - 6\mathcal{P}(z; 0, \alpha_1)\hat{A}_7 = 72 \left(\hat{A}_1 \hat{A}_3 \hat{A}_1' + \hat{A}_1^2 \hat{A}_3' \right). \quad (3.14)$$

So the homogenous solution of the above equation is nothing but the Weierstrass elliptic function $\mathcal{P}(z; 0, \alpha_1)$. Therefore the compatibility conditions (3.4) allow one to determine all the nonzero coefficients $A_k(z)$ in terms of the Weierstrass elliptic function $\mathcal{P}(z)$.

I.b: $a_1 \neq 0$: Equation (3.2a) implies that $r_1 r_2 r_3 = 24$. Under this condition there are four possible cases of (r_1, r_2, r_3) such that $r_i > 0$ and distinct integers, but there is only the following case out of the four cases such that the compatibility conditions at the resonances for the simplified equations are identically satisfied and $y_0 \neq 0$

$$(r_1, r_2, r_3) = (2, 3, 4), \quad y_0 = -2/a_1, \quad a_2 = 3a_1, \quad a_3 = a_4 = 0. \quad (3.15)$$

By adding the nondominant terms to the simplified equation, using the linear transformation (2.9) and the compatibility conditions one finds the canonical form of the equation

$$\begin{aligned}
y^{(4)} &= -2yy''' - 6y'y'' + A_1(y''' + 2yy'' + 2y'^2) \\
&\quad + A_3(y'' + 2yy') + A_7(y' + y^2) + A_{12}, \quad (3.16)
\end{aligned}$$

where A_1 , A_3 , A_7 and A_{12} are arbitrary functions of z . If one lets $u = y^2 + y'$, then the equation (3.16) can be linearized. Equation (3.16) was considered in [3, 9].

Case II. $a_5 = a_6 = 0$: In this case there are two branches corresponding to $(-1, y_{0j})$, $j = 1, 2$, where y_{0j} are the roots of (3.2.b) and

$$y_{01} + y_{02} = \frac{2(3a_1 + a_2)}{2a_3 + a_4}, \quad y_{01}y_{02} = -\frac{24}{2a_3 + a_4}. \quad (3.17)$$

Let (r_{j1}, r_{j2}, r_{j3}) be the roots (additional to $r_0 = -1$) of the resonance equation (3.2a) corresponding to y_{0j} . When one sets

$$P(y_{0j}) = -2(2a_3 + a_4)y_{0j}^2 + 6(3a_1 + a_2)y_{0j} + 96, \quad j = 1, 2, \quad (3.18)$$

(3.2a) implies that

$$\prod_{i=1}^3 r_{ji} = P(y_{0j}) = p_j, \quad j = 1, 2, \quad (3.19)$$

where the p_j are integers and at least one of them is a positive integer in order to have the principal branch. Let the branch corresponding to y_{01} be the principal branch, that is $p_1 > 0$. Equations (3.17) and (3.18) give

$$P(y_{01}) = 24 \left(1 - \frac{y_{01}}{y_{02}} \right) = p_1, \quad P(y_{02}) = 24 \left(1 - \frac{y_{02}}{y_{01}} \right) = p_2. \quad (3.20)$$

Hence the p_j satisfy the following Diophantine equation, if $p_1 p_2 \neq 0$,

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{24}. \quad (3.21)$$

There are 21 integer solutions (p_1, p_2) of (3.21) such that one of the p_j is positive. Once p_1 is known, for each p_1 one can write possible distinct positive integers (r_{11}, r_{12}, r_{13}) such that $\prod_{i=1}^3 r_{1i} = p_1$. Then for each set of (r_{11}, r_{12}, r_{13}) , a_k , $k = 2, 3, 4$, and y_{0j} can be determined in terms of a_1 by using

$$\sum_{i=1}^3 r_{ji} = 11 + a_1 y_{0j}, \quad \sum_{i \neq k} r_{ji} r_{jk} = -a_3 y_{0j}^2 + (7a_1 + a_2)y_{0j} + 46, \quad (3.22)$$

for $j = 1$, and the equation (3.17). Then for these values of a_k and y_{0j} one should find all cases such that the resonance equation (3.2a) has distinct integral roots r_{2i} corresponding to y_{02} . Only for the following cases **a)** $(p_1, p_2) = (12, -24)$ and **b)** $(p_1, p_2) = (20, -120)$ are all the resonances distinct integers for both branches, one of which is the principal branch. The resonances and the simplified equations for these cases are as follows:

II.a: $(p_1, p_2) = (12, -24)$:

$$\begin{aligned} y_{01} &= -\frac{3}{a_1} : (r_{11}, r_{12}, r_{13}) = (1, 3, 4), \\ y_{02} &= -\frac{6}{a_1} : (r_{21}, r_{22}, r_{23}) = (-2, 3, 4), \\ y^{(4)} &= a_1 \left(yy''' + 3y'y'' - \frac{1}{3}a_1 y^2 y'' - \frac{2}{3}a_1 y y'^2 \right), \end{aligned} \quad (3.23)$$

II.b: $(p_1, p_2) = (20, -120)$:

$$\begin{aligned} y_{01} &= -\frac{1}{a_1} : (r_{11}, r_{12}, r_{13}) = (1, 4, 5), \\ y_{02} &= -\frac{6}{a_1} : (r_{21}, r_{22}, r_{23}) = (-5, 4, 6), \\ y^{(4)} &= a_1 (yy''' + 11y'y'' - a_1 y^2 y'' - 2a_1 y y'^2). \end{aligned} \quad (3.24)$$

For case II.a the compatibility conditions at the resonances of the simplified equation are identically satisfied. For the case II.b the compatibility condition at the resonance $r_{13} = 5$ implies that $y_4 = 0$ which contradicts with the arbitrariness of y_4 . Moreover in the case II.b, if one lets $y = \lambda u$ such that $\lambda a_1 = 1$, integration of the simplified equation once yields

$$u''' = uu'' + 5u'^2 - u^2u' + c, \quad (3.25)$$

where c is an arbitrary integration constant. Equation (3.25) is not a Painlevé type equation unless $c = 0$. It was studied in [3, 17]. Hence we consider case II.a. Adding the nondominant terms to the simplified equation and by using the linear transformation (2.9) and the compatibility conditions of the first branch we can determine the coefficients $A_k(z)$ of the nondominant terms. The canonical form of the equation for the case II.a is

$$\begin{aligned} y^{(4)} = & -3yy''' - 9y'y'' - 3y^2y'' - 6yy'^2 + Ry'' \\ & + 2R'y' + R''y + A_9(y^3 + 3yy' + y'' - Ry) + A_{12}, \end{aligned} \quad (3.26)$$

where $R(z) = A_3(z) - A_9(z)$ and A_3 and A_9 are arbitrary analytic functions of z . If one lets

$$u = y'' + 3yy' + y^3 - Ry, \quad (3.27)$$

then equation (3.26) can be reduced to a linear equation for u . Equation (3.26) was considered in [3, 9].

Case III. $a_6 = 0$: There are three branches corresponding to y_{0j} , $j = 1, 2, 3$, which are the roots of the equation (3.2b). If one lets

$$\begin{aligned} \prod_{i=1}^3 r_{ji} = p_j = & P(y_{0j}) = a_5 y_{0j}^3 - 2(2a_3 + a_4) y_{0j}^2 \\ & + 6(3a_1 + a_2) y_{0j} + 96, \quad j = 1, 2, 3, \end{aligned} \quad (3.28)$$

where p_j are integers and at least one of them is positive, by the use of the same procedure as was carried in the previous case the p_j satisfy the following Diophantine equation:

$$\sum_{j=1}^3 \frac{1}{p_j} = \frac{1}{24}, \quad (3.29)$$

if $p_1 p_2 p_3 \neq 0$ and, if $a_1 \neq 0$,

$$\prod_{j=1}^3 p_j = -\frac{24^3}{(y_{01} y_{02} y_{03})^2} (y_{01} - y_{02})^2 (y_{01} - y_{03})^2 (y_{02} - y_{03})^2. \quad (3.30)$$

Let $p_1, p_2 > 0$ and $p_3 < 0$. If (r_{j1}, r_{j2}, r_{j3}) are the resonances corresponding to y_{0j} respectively, then they satisfy equation (3.22) for $j = 1, 2, 3$. There are the following two cases which should be considered separately.

III.a: $a_1 = 0$: Equation (3.22a) for $j = 1$ implies that there are five possible values of (r_{11}, r_{12}, r_{13}) and hence five possible values of p_1 . For each value of p_1 one can solve (3.29) such that $p_2 > 0$, $p_3 < 0$ and both are integers. Then for each (p_1, p_2, p_3) the equations

$$\begin{aligned} p_1 &= 24 \left(1 - \frac{y_{01}}{y_{02}}\right) \left(1 - \frac{y_{01}}{y_{03}}\right), & p_2 &= 24 \left(1 - \frac{y_{02}}{y_{01}}\right) \left(1 - \frac{y_{02}}{y_{03}}\right), \\ p_1 &= 24 \left(1 - \frac{y_{03}}{y_{01}}\right) \left(1 - \frac{y_{03}}{y_{02}}\right), \end{aligned} \quad (3.31)$$

give the equations (2.58) for y_{0j} for

$$k = \frac{24}{y_{01}y_{02}y_{03}}(y_{01} - y_{02})(y_{02} - y_{03})(y_{01} - y_{03}). \quad (3.32)$$

The system (2.58) has nontrivial solution if $k^2 = -(p_1p_2 + p_1p_3 + p_2p_3)$. For each value of k one can find y_{0j} and a_i , $i = 3, 4, 5$, in terms of a_2 . Once the coefficients of the resonance equation (3.2a) are known for all branches, one should look at the cases such that the roots of (3.2a) are distinct integers for the second and third branches. There is only one case, $(p_1, p_2, p_3) = (40, 40, -120)$, and $k = 40\sqrt{5}$. The y_{0j} , the resonances and the simplified equation for this case are as follows:

$$\begin{aligned} y_{01} &= \frac{4}{a_2} (1 - \sqrt{5}) : (r_{11}, r_{12}, r_{13}) = (2, 4, 5), \\ y_{02} &= \frac{4}{a_2} (1 + \sqrt{5}) : (r_{21}, r_{22}, r_{23}) = (2, 4, 5), \\ y_{03} &= \frac{24}{a_2} : (r_{31}, r_{32}, r_{33}) = (-3, 4, 10), \\ y^{(4)} &= a_2 \left(y'y'' + \frac{1}{8}a_2y^2y'' + \frac{1}{4}a_2yy'^2 + \frac{1}{64}a_2^2y^3y' \right). \end{aligned} \quad (3.33)$$

The compatibility conditions are identically satisfied for the simplified equation. To obtain the canonical form of the equation one should add the nondominant terms with analytic coefficients $A_k(z)$, $k = 1, \dots, 12$. The linear transformation (2.9) and the compatibility conditions at the resonances of the first and second branches give the following equation

$$y^{(4)} = 24y'y'' + 72y^2y'' + 144yy'^2 + 216y^3y'^2. \quad (3.34)$$

Integration of (3.34) once gives (2.61).

III.b: $a_1 \neq 0$: In this case the resonances (r_{j1}, r_{j2}, r_{j3}) and y_{0j} satisfy (3.22) for $j = 1, 2, 3$ and

$$\sum_{i=1}^3 y_{0j} = \frac{1}{a_5}(2a_3 + a_4), \quad \sum_{j \neq k} y_{0j}y_{0k} = -\frac{2}{a_5}(3a_1 + a_2), \quad \prod_{i=1}^3 y_{0j} = -\frac{24}{a_5}, \quad (3.35)$$

respectively. The $p_j = \prod_{j=1}^3 r_{ji}$ satisfy the Diophantine equation (3.29). If one lets

$$n^2 = \frac{24^2}{(y_{01}y_{02}y_{03})^2}(y_{01} - y_{02})^2(y_{01} - y_{03})^2(y_{02} - y_{03})^2, \quad (3.36)$$

then (3.30) gives

$$p_1 p_2 \hat{p}_3 = 24n^2, \quad (3.37)$$

where $\hat{p}_3 = -p_3$ and $p_1 < 48$. If one follows the procedure given in the previous section, (3.29) and (3.37) give that

$$(p_1 \hat{p}_3)^2 = n^2 [24p_1 - (24 - p_1) \hat{p}_3], \quad (p_1 p_2)^2 = n^2 [24p_1 + (24 - p_1) p_2], \quad (3.38)$$

for $p_1 < 24$ and for $24 < p_1 < 48$ respectively. So the right hand sides of both equations in (3.38) must be complete squares. Based on these conditions on p_i , $i = 1, 2, 3$, there are 71 integer solutions (p_1, p_2, p_3) of the Diophantine equation (3.29). For each solution (p_1, p_2, p_3) , one can find y_{0j} by solving the system of equations (2.58). Then one can write possible resonances (r_{11}, r_{12}, r_{13}) for each p_1 provided that

$$a_1 y_{01} = \sum_{i=1}^3 r_{1i} - 11 \quad (3.39)$$

are all integers. There are the following three cases such that all the resonances of all three branches are distinct integers.

III.b.i: $(p_1, p_2, p_3) = (15, 60, -24)$:

$$\begin{aligned} y_{01} &= -\frac{2}{a_1} : (r_{11}, r_{12}, r_{13}) = (1, 3, 5), \\ y_{02} &= -\frac{12}{a_1} : (r_{21}, r_{22}, r_{23}) = (-2, -5, 6), \\ y_{03} &= -\frac{8}{a_1} : (r_{31}, r_{32}, r_{33}) = (-4, 1, 6), \\ a_2 &= \frac{11}{2} a_1, \quad a_3 = -\frac{1}{2} a_1^2, \quad a_4 = -\frac{7}{4} a_1^2, \quad a_5 = \frac{1}{8} a_1^3. \end{aligned} \quad (3.40)$$

III.b.ii: $(p_1, p_2, p_3) = (24, n, -n)$, $n > 0$, $n \neq 24$:

$$\begin{aligned} y_{01} &= -\frac{2}{a_1} : (r_{11}, r_{12}, r_{13}) = (2, 3, 4), \\ y_{02} &= -\frac{1}{a_1} \left(1 - \frac{n}{24}\right) : (r_{21}, r_{22}, r_{23}) = \left(4, 6, \frac{n}{24}\right), \\ y_{03} &= -\frac{1}{a_1} \left(1 + \frac{n}{24}\right) : (r_{31}, r_{32}, r_{33}) = \left(4, 6, -\frac{n}{24}\right), \\ a_2 &= \frac{15552 - 3n^2}{576 - n^2} a_1, \quad a_3 = -\frac{6912}{576 - n^2} a_1^2, \\ a_4 &= -\frac{13824}{576 - n^2} a_1^2, \quad a_5 = \frac{6912}{576 - n^2} a_1^3. \end{aligned} \quad (3.41)$$

III.b.iii: $(p_1, p_2, p_3) = (24, n, -n)$, $n > 0$, $n \neq 4, 24$:

$$\begin{aligned} y_{01} &= -\frac{12}{a_1} : (r_{11}, r_{12}, r_{13}) = (-2, -3, 4), \\ y_{02} &= -\frac{1}{a_1} \left(6 - \frac{n}{4}\right) : (r_{21}, r_{22}, r_{23}) = \left(1, 4, \frac{n}{4}\right), \\ y_{03} &= -\frac{1}{a_1} \left(6 + \frac{n}{4}\right) : (r_{31}, r_{32}, r_{33}) = \left(1, 4, -\frac{n}{4}\right), \end{aligned}$$

$$\begin{aligned} a_2 &= \frac{1152 + 2n^2}{576 - n^2} a_1, & a_3 &= -\frac{192}{576 - n^2} a_1^2, \\ a_4 &= -\frac{384}{576 - n^2} a_1^2, & a_5 &= \frac{32}{576 - n^2} a_1^3. \end{aligned} \quad (3.42)$$

For all three cases the simplified equations pass the Painlevé test. To obtain the canonical form of the equation one should add the nondominant terms with the coefficients $A_k(z)$, $k = 1, \dots, 12$. The linear transformation (2.9) and the compatibility conditions at the resonances give the following equations:

III.b.i:

$$\begin{aligned} y^{(4)} &= -2yy''' - 11y'y'' - 2y^2y'' - 7yy'^2 - y^3y'^2 + A_6(y'' + yy') \\ &\quad + \frac{1}{3}A_6'(y^2 + 4y') + \frac{1}{3}A_6''' - \frac{2}{9}A_6A_6', \end{aligned} \quad (3.43)$$

where A_6 is an arbitrary function of z . Equation (3.43) was given in [9].

III.b.ii: The compatibility condition at the resonance $r = 6$ for the third branch gives

$$A_1' + A_1^2 = 0. \quad (3.44)$$

So following two subcases should be considered separately.

III.b.ii.1: $A_1 = 0$: The canonical form of the equation is

$$\begin{aligned} y^{(4)} &= -2yy''' - \frac{6}{m^2 - 1} [(m^2 - 9)y'y'' - 8y^2y'' - 16(yy'^2 + y^3y')] \\ &\quad + A_3(y'' + 2yy') + (A_3' + c_1)(y' + y^2) + A_{12}, \end{aligned} \quad (3.45)$$

where $m = n/24$, $m \neq 1, 4, 6$, A_3 is an arbitrary function of z and

$$A_{12} = \frac{m^2 - 1}{48} (A_3''' - A_3A_3' - c_1A_3 + 2c_1^2z + c_2), \quad c_1, c_2 = \text{constant}. \quad (3.46)$$

The result (3.45) was given in [9].

III.b.ii.2: $A_1 = 1/(z - c)$: Without loss of generality one can set $c = 0$. The canonical form of the equation is

$$\begin{aligned} y^{(4)} &= -2yy''' + \frac{1}{m^2 - 1} [(54 - 6m^2)y'y'' + 48y^2y'' + 96(yy'^2 + y^3y')] \\ &\quad + \frac{1}{z} \left\{ y''' + 2yy'' - \frac{1}{m^2 - 1} [(26 - 2m^2)y'^2 + 48y^2y' + 24y^4] \right\} \\ &\quad + A_3(y'' + 2yy') + \left(A_3' - A_3 \frac{1}{z} + c_1z \right) (y' + y^2) + A_{12}, \end{aligned} \quad (3.47)$$

where A_3 is an arbitrary function of z and

$$A_{12} = -\frac{m^2 - 1}{48} \left(A_3''' - \frac{1}{z}A_3'' - A_3A_3' + \frac{1}{2z}A_3^2 - c_1zA_3 + \frac{1}{2}c_1^2z^3 \right) + \frac{c_2}{z}, \quad (3.48)$$

where c_1 and c_2 are constants. The result (3.47) was given in [9].

III.b.iii: If we let $m = n/4$, $m \neq 1, 4, 6$, the canonical form of the equation for $m = 2$ is

$$y^{(4)} = 2yy''' + 5y'y'' - \frac{3}{2}y^2y'' - 3yy'^2 + \frac{1}{2}y^3y' + A_1 \left[y''' - 2yy'' - \frac{3}{2}y'^2 + \frac{3}{2}y^2y' - \frac{1}{8}y^4 - A_7y \right] + A_7y' + A_7'y + A_{12}. \quad (3.49)$$

If one sets

$$u = y''' - 2yy'' - \frac{3}{2}y'^2 + \frac{3}{2}y^2y' - \frac{1}{8}y^4 - A_7y, \quad (3.50)$$

then (3.49) can be reduced to a linear equation for u . It should be noted that (3.50) belongs to $P_{\Pi}^{(3)}$ and was given in (2.75). For $m = 3$

$$y^{(4)} = 2yy''' + \frac{20}{3}y'y'' - \frac{16}{9}y^2y'' - \frac{32}{9}yy'^2 + \frac{16}{27}y^3y' + A_1 \left(y''' - 2yy'' - \frac{7}{3}y'^2 + \frac{16}{9}y^2y' - \frac{4}{27}y^4 \right) + A_{12}, \quad (3.51)$$

where A_1 and A_{12} are arbitrary functions of z . If one sets

$$u = y''' - 2yy'' - \frac{7}{3}y'^2 + \frac{16}{9}y^2y' - \frac{4}{27}y^4, \quad (3.52)$$

(3.51) can be reduced to a linear equation in u . Equation (3.52) belongs to $P_{\Pi}^{(3)}$ and was given in (2.76). Equations (3.49) and (3.51) were given in [9]. It should be noted, that for $m \geq 4$, integration of the simplified equation once gives the simplified equation of the case given in (2.74) with an additional integration constant c . Thus for $m \geq 4$ the simplified equation is not of Painlevé type if $c \neq 0$.

Case IV. $a_6 \neq 0$: In this case there are four branches corresponding to $(-1, y_{0j})$, $j = 1, 2, 3, 4$. If (r_{j1}, r_{j2}, r_{j3}) are the resonances corresponding to the branches, $\prod_{i=1}^3 r_{ji} = p_j$ such that the p_j are integers and at least one of them is positive. Then (3.2a) implies that

$$P(y_{0j}) = a_5y_{0j}^3 - 2(2a_3 + a_4)y_{0j}^2 + 6(3a_1 + a_2)y_{0j} + 96 = p_j, \quad j = 1, 2, 3, 4. \quad (3.53)$$

On the other hand (3.2b) implies that

$$\begin{aligned} \sum_{j=1}^4 y_{0j} &= \frac{a_5}{a_6}, & \sum_{j \neq i} y_{0j} y_{0i} &= \frac{2a_3 + a_4}{a_6}, \\ \sum_{j \neq i \neq k} y_{0j} y_{0i} y_{0k} &= \frac{2(3a_1 + a_2)}{a_6}, & \prod_{j=1}^4 y_{0j} &= -\frac{24}{a_6}. \end{aligned} \quad (3.54)$$

Then (3.53) yields

$$p_j = P(y_{0j}) = 24 \prod_{j \neq k} \left(1 - \frac{y_{0j}}{y_{0k}} \right), \quad j = 1, 2, 3, 4. \quad (3.55)$$

Therefore the p_j satisfy the Diophantine equation

$$\sum_{j=1}^4 \frac{1}{p_j} = \frac{1}{24}. \quad (3.56)$$

To find the simplified equation one should follow the following steps: **a)** Find all integer solutions (p_1, p_2, p_3, p_4) of the Diophantine equation (3.56). **b)** For each pair (p_1, p_2) from the solution set of the Diophantine equation, write all possible (r_{j1}, r_{j2}, r_{j3}) such that $\prod_{i=1}^3 r_{ji} = p_j$, $j = 1, 2$. **c)** Determine y_{01} and y_{02} in terms of a_1 , if $a_1 \neq 0$, by using the equation (3.22a) for $j = 1, 2$. **d)** Use (3.55) to find y_{03} and y_{04} in terms of a_1 . **e)** Eliminate the cases (r_{j1}, r_{j2}, r_{j3}) $j = 1, 2$, such that $a_1 y_{0k}$, $k = 3, 4$, are not integers (see the equation (3.22a)). **f)** Find a_i , $i = 2, \dots, 6$, in terms of a_1 by using the (3.53) and (3.54). Once all the coefficients of the equation (3.2a) are known, look at the cases such that the roots of (3.2a) are distinct integers for y_{03} and y_{04} .

There are four cases such that all the resonances are distinct integers for all branches. These cases and the corresponding simplified equations are as follows:

IV.a: $(p_1, p_2, p_3, p_4) = (6, -4, 6, -24)$:

$$\begin{aligned} y_{01} &= -\frac{5}{a_1} : (r_{11}, r_{12}, r_{13}) = (1, 2, 3), \\ y_{02} &= -\frac{10}{a_1} : (r_{21}, r_{22}, r_{23}) = (-2, 1, 2), \\ y_{03} &= -\frac{15}{a_1} : (r_{31}, r_{32}, r_{33}) = (-3, -2, 1), \\ y_{04} &= -\frac{20}{a_1} : (r_{41}, r_{42}, r_{43}) = (-4, -3, -2), \\ y^{(4)} &= a_1 \left(yy''' + 2y'y'' - \frac{2}{5}a_1 y^2 y'' - \frac{3}{5}a_1 y y'^2 + \frac{2}{25}a_1^2 y^3 y' - \frac{1}{625}a_1^3 y^5 \right), \end{aligned} \quad (3.57)$$

IV.b: $(p_1, p_2, p_3, p_4) = (36, 36, -84, -504)$:

$$\begin{aligned} y_{01} &= -\frac{5}{a_2} : (r_{11}, r_{12}, r_{13}) = (2, 3, 6), \\ y_{02} &= \frac{10}{a_2} : (r_{21}, r_{22}, r_{23}) = (2, 3, 6), \\ y_{03} &= \frac{15}{a_2} : (r_{31}, r_{32}, r_{33}) = (-2, 6, 7), \\ y_{04} &= -\frac{20}{a_2} : (r_{41}, r_{42}, r_{43}) = (-7, 6, 12), \\ y^{(4)} &= a_2 \left[y'y'' + \frac{1}{5}a_2 \left(y^2 y'' + yy'^2 - \frac{1}{125}a_2^2 y^5 \right) \right], \end{aligned} \quad (3.58)$$

IV.c: $(p_1, p_2, p_3, p_4) = (36, 36, -144, -144)$:

$$\begin{aligned} y_{01}^2 &= \frac{10}{a_3} : (r_{11}, r_{12}, r_{13}) = (2, 3, 6), \\ y_{02} &= -y_{01} : (r_{21}, r_{22}, r_{23}) = (2, 3, 6), \end{aligned}$$

$$\begin{aligned}
y_{03}^2 &= \frac{40}{a_3} : (r_{31}, r_{32}, r_{33}) = (-3, 6, 8), \\
y_{04} &= -y_{03} : (r_{41}, r_{42}, r_{43}) = (-3, 6, 8), \\
y^{(4)} &= a_3 \left(y^2 y'' + y y'^2 - \frac{3}{50} a_3 y^5 \right), \tag{3.59}
\end{aligned}$$

IV.d: $(p_1, p_2, p_3, p_4) = (20, -120, -60, 60) :$

$$\begin{aligned}
y_{01} &= \frac{2}{a_1} : (r_{11}, r_{12}, r_{13}) = (1, 2, 10), \\
y_{02} &= -\frac{8}{a_1} : (r_{21}, r_{22}, r_{23}) = (-10, 1, 12), \\
y_{03} &= \frac{4}{a_1} : (r_{31}, r_{32}, r_{33}) = (-2, 2, 15), \\
y_{04} &= -\frac{6}{a_1} : (r_{41}, r_{42}, r_{43}) = (-3, -2, 10), \\
y^{(4)} &= a_1 \left(yy''' - \frac{17}{2} y' y'' + \frac{11}{4} a_1 y^2 y'' - \frac{15}{4} a_1 y y'^2 + \frac{1}{2} a_1^2 y^3 y' - \frac{1}{16} a_1^3 y^5 \right). \tag{3.60}
\end{aligned}$$

The simplified equation for the case IV.d does not pass the Painlevé test. So this case is not be considered. The canonical forms for the other cases can be obtained by adding the nondominant terms with the coefficients $A_k(z)$, $k = 1, \dots, 12$ to the simplified equations. All coefficients A_k can be obtained by using the linear transformation (2.9) and the compatibility conditions at the resonances. The canonical forms are as follows:

IV.a:

$$\begin{aligned}
y^{(4)} &= -5yy''' - 10(y'y'' + y^2y'' + y^3y') - 15yy'^2 - y^5 + A_1(y''' + 4yy'' + 3y'^2 \\
&\quad + 6y^2y' + y^4) + A_3(y'' + 3yy' + y^3) + A_7(y' + y^2) + A_{11}y + A_{12}. \tag{3.61}
\end{aligned}$$

If one lets $y = u'/u$, (3.61) gives the fifth order linear equation for u . Equation (3.61) was given in [9].

IV.b:

$$y^{(4)} = -5y'y'' + 5y^2y'' + 5yy'^2 - y^5 + (c_1z + c_2)y + c_3, \tag{3.62}$$

where c_i are constants. The result (3.62) was given in [9].

IV.c:

$$y^{(4)} = 10y^2y'' + 10yy'^2 - 6y^5 + c_1(y'' - 2y^3) + (c_2z + c_3)y + c_4, \tag{3.63}$$

where c_i are constants. The result (3.63) was given in [9, 16].

4 Fifth order equations: $P_{II}^{(5)}$

Differentiation of (3.1) and addition of the term y^6 which is also of order -6 as $z \rightarrow z_0$ gives the following simplified equation of order five

$$\begin{aligned}
y^{(5)} &= a_1yy^{(4)} + a_2y'y''' + a_3y''^2 + a_4y^2y''' + a_5yy'y'' + a_6y'^3 \\
&\quad + a_7y^3y'' + a_8y^2y'^2 + a_9y^4y' + a_{10}y^6, \tag{4.1}
\end{aligned}$$

where a_i , $i = 1, \dots, 10$, are constants. Substitution of (2.4) into (4.1) into the equation above gives the following equations for resonance r and for y_0 respectively,

$$\begin{aligned} Q(r) = (r+1) \Big\{ & r^4 - (16 + a_1 y_0) r^3 - [a_4 y_0^2 - (11a_1 + a_2) y_0 - 101] r^2 \\ & - [a_7 y_0^3 - (7a_4 + a_5) y_0 + (46a_1 + 7a_2 + 4a_3) y_0 + 326] r \\ & - [a_9 y_0^4 - 2(2a_7 + a_8) y_0^3 + 3(6a_4 + 2a_5 + a_6) y_0^2 \\ & - 8(12a_1 + 3a_2 + 2a_3) y_0 - 600] \Big\} = 0, \end{aligned} \quad (4.2)$$

$$\begin{aligned} a_{10} y_0^5 - a_9 y_0^4 + (2a_7 + a_8) y_0^3 - (6a_4 + 2a_5 + a_6) y_0^2 \\ + 2(12a_1 + 3a_2 + 2a_3) y_0 + 120 = 0. \end{aligned} \quad (4.3)$$

Equation (4.3) implies that there are five branches if $a_6 \neq 0$. If $(r_{j1}, r_{j2}, r_{j3}, r_{j4})$, $j = 1, \dots, 5$, are the distinct integer resonances corresponding to $(-1, y_{0j})$ and if $\prod_{i=1}^4 r_{ji} = p_j$, where p_j are integers and at least one of them is positive, then the p_j satisfy the following Diophantine equation,

$$\sum_{j=1}^5 \frac{1}{p_j} = \frac{1}{120}. \quad (4.4)$$

The determination of all integer solutions $(p_1, p_2, p_3, p_4, p_5)$ of the Diophantine equation is quite difficult. So, for the sake of completeness, in this section we present special cases such as single, double and triple branch cases. We also give an example for the case of the four branches. Since the procedure to obtain the canonical form of the differential equations is the same as described in the previous sections, we only give the canonical form of the differential equations for each cases.

The canonical form of the equation can be obtained by adding the nondominant terms $y^{(4)}$, yy''' , $y'y''$, y^2y'' , yy'^3 , y^3y' , y^5 , y''' , yy'' , y'^2 , y^2y' , y^4 , y'' , yy' , y^3 , y' , y^2 , y , 1 with coefficients $A_k(z)$, $k = 1, \dots, 19$, which are analytic functions of z , respectively.

Case I. If $a_l = 0$, $l = 4, \dots, 10$, there is only one branch and there are two cases such that the resonances are distinct positive integers.

I.a:

$$\begin{aligned} y_{01} = -2/a_1 : (r_{11}, r_{12}, r_{13}, r_{14}) &= (2, 3, 4, 5), \\ y^{(5)} = -2yy^{(4)} - 8y'y''' - 6y''^2 + A_1(y^4 + 2yy'' + 6y'y'') &+ A_8(y'' + 2yy'' + 2y'^2) \\ &+ A_{13}(y'' + 2yy') + A_{16}(y' + y^2) + A_{19}, \end{aligned} \quad (4.5)$$

where $A_1, A_8, A_{13}, A_{16}, A_{19}$ are arbitrary analytic functions of z . Equation (4.5) can be linearized by letting $u = y' + y^2$.

I.b:

$$y_{01} = -12/a_2 : (r_{11}, r_{12}, r_{13}, r_{14}) = (1, 4, 5, 6).$$

In this case the linear transformation and the compatibility conditions give $A_i = 0$, $i = 1, \dots, 7$, $A_{11} = A_{12} = A_{15} = 0$ and

$$A_9'' - \frac{1}{2}A_9^2 = 0. \quad (4.6)$$

Depending on the solution of (4.6) there are following two subcases.

I.b.i: $A_9 = 0$. The canonical form of the equation is

$$y^{(5)} = -12y'y''' - 12y''^2 + (c_1z + c_2)y'' + 2c_1y' + \frac{1}{6}(c_1z + c_2)^2, \quad (4.7)$$

where c_1 and c_2 are constants. If $c_1 \neq 0$, (4.7) can be reduced to (2.14). If $c_1 = 0$, (4.7) can be reduced to a third order equation which belongs to the hierarchy of the first Painlevé equation, $P_I^{(3)}$ [18], by integration once and letting $y = u'$.

I.b.ii: $A_9 = 12/z^2$. The canonical form of the equation is

$$\begin{aligned} y^{(5)} = & -12y'y''' - 12y''^2 + \frac{12}{z^2} \left(\frac{3}{2}y''' + yy'' + 2y'^2 \right) + \left(c_1z^3 + \frac{c_2}{z^2} - \frac{24}{z^3} \right) y''' \\ & - \frac{48}{z^3} yy' + \left(6c_1z^2 - \frac{4c_2}{z^3} + \frac{24}{z^4} \right) y' + \left(4c_1 + \frac{4c_2}{z^4} \right) y \\ & + \frac{24}{z^4} y^2 + \frac{1}{6} \left(c_1z^3 + \frac{c_2}{z^2} \right)^2, \end{aligned} \quad (4.8)$$

where c_1 and c_2 are constants.

Case II. $a_7 = \dots = a_{10} = 0$: In this case there are two branches. The resonances and the canonical form of the equation are

$$\begin{aligned} y_{01} = \frac{-3}{a_1} : (r_{11}, r_{12}, r_{13}, r_{14}) &= (1, 3, 4, 5), \\ y_{02} = \frac{-6}{a_1} : (r_{21}, r_{22}, r_{23}, r_{24}) &= (-2, 3, 4, 5), \\ y^{(5)} = & -3yy^{(4)} - 12y'y''' - 9y''^2 - 18yy'y'' - 6y'^3 - 3y^2y'' + (Ry)''' \\ & + \frac{1}{3}A_{10} [y''' + 3yy'' + 3y^2y' + 3y'^2 - (Ry)'] \\ & + A_{15} (y'' + 3yy' + y^3 - Ry) + A_{19}, \end{aligned} \quad (4.9)$$

where $R = A_8 - A_9/3$ and A_8, A_9, A_{10}, A_{15} and A_{19} are arbitrary analytic functions of z . Equation (4.9) can be linearized if one lets

$$u = y'' + 3yy' + y^3 - Ry. \quad (4.10)$$

Case III. $a_9 = a_{10} = 0$: In this case there are three branches. The resonances and the canonical form of the equations are as follows:

III.a:

$$\begin{aligned} y_{01} = \frac{-2}{a_1} : (r_{11}, r_{12}, r_{13}, r_{14}) &= (2, 3, 4, 5), \\ y_{02} = -\frac{1-n}{a_1} : (r_{21}, r_{22}, r_{23}, r_{24}) &= (4, 5, 6, n), \\ y_{03} = -\frac{1+n}{a_1} : (r_{31}, r_{32}, r_{33}, r_{34}) &= (4, 5, 6, -n), \\ y^{(5)} = & -2yy^{(4)} + \frac{1}{n^2-1} [(56-8n^2)y'y''' + (54-6n^2)y''^2 + 48y^2y'' \\ & + 288yy'y'' + 96(y'^3 + y^3y'') + 288y^2y'^2] + A_8(y''' + 2yy'' + y'^2) \\ & + (2A'_8 + c_1z + c_2)(y'' + 2yy') + (A''_8 + 2c_1)(y' + y^2) + A_{19}, \end{aligned} \quad (4.11)$$

where A_8 is an arbitrary analytic function of z and

$$A_{19} = -\frac{n^2 - 1}{48} [A_8''' - A_8 A_8'' - A_8'^2 - A_8'(c_1 z + c_2) - 2c_1 A_8 + 2(c_1 z + c_2)^2], \quad (4.12)$$

and c_1, c_2 are constants, $n \in \mathbb{Z}_+$, $n \neq 1, 4, 5, 6$. If $c_1 = c_2 = 0$, double integration of (4.11) yields (2.77).

III.b: The resonances are

$$\begin{aligned} y_{01} &= -\frac{6-n}{a_1} : (r_{11}, r_{12}, r_{13}, r_{14}) = (1, 4, 5, n), \\ y_{02} &= -\frac{6+n}{a_1} : (r_{21}, r_{22}, r_{23}, r_{24}) = (1, 4, 5, -n), \\ y_{03} &= -\frac{12}{a_1} : (r_{31}, r_{32}, r_{33}, r_{34}) = (-3, -2, 4, 5), \end{aligned} \quad (4.13)$$

where $n \in \mathbb{Z}_+$ and $n \neq 1, 4, 5$. It should be noted that, when $n \geq 6$, the double integration of the simplified equation gives the third order equation (2.73) with an additional term $(c_1 z + c_2)$. Therefore the simplified equation is not of Painlevé type if c_1 and $c_2 \neq 0$. Hence we only consider the cases for $n = 2, 3$. The canonical form of the equation for $n = 2$ is

$$\begin{aligned} y^{(5)} &= 2yy^{(4)} + 7y'y''' + 5y''^2 - \frac{3}{2}y^2y''' - 9yy'y'' - 3y'^3 + \frac{1}{2}y^3y'' \\ &\quad + \frac{3}{2}y^2y'^2 + A_1 \left(y^{(4)} - 2yy''' - 5y'y'' + \frac{3}{2}y^2y' + 3yy'^2 - \frac{1}{2}y^3y' \right) \\ &\quad + A_8 \left(y''' - 2yy'' - \frac{3}{2}y'^2 + \frac{3}{2}y^2y' - \frac{1}{8}y^4 \right) + A_{13}y'' \\ &\quad + (2A_{13}' - A_1 A_{13})y' + (A_{13}'' - A_1 A_{13}' - A_8 A_{13})y + A_{19}, \end{aligned} \quad (4.14)$$

where A_1, A_8, A_{13} and A_{19} are arbitrary analytic functions of z . Double integration of (4.14) yields (2.75).

For $n = 3$

$$\begin{aligned} y^{(5)} &= 2yy^{(4)} + \frac{1}{3} \left(26y'y''' + 20y''^2 - \frac{16}{3}y^2y''' - 32yy'y'' - \frac{32}{3}y'^3 + \frac{16}{9}y^3y'' \right. \\ &\quad \left. + \frac{16}{3}y^2y'^2 \right) + A_1 \left[y^{(4)} - 2yy''' - \frac{2}{3} \left(10y'y'' - \frac{8}{3}y^2y'' - \frac{16}{3}yy'^2 + \frac{8}{9}y^3y' \right) \right] \\ &\quad + A_8 \left(y''' - 2yy'' - \frac{7}{3}y'^2 + \frac{16}{9}y^2y' - \frac{4}{27}y^4 \right) + A_{19}, \end{aligned} \quad (4.15)$$

where A_1, A_8 and A_{19} are arbitrary analytic functions of z . If one lets

$$u = y''' - 2yy'' - \frac{7}{3}y'^2 + \frac{16}{9}y^2y' - \frac{4}{27}y^4, \quad (4.16)$$

(4.15) can be reduced to a linear equation for u . It should be noted that (4.16) belongs to $P_{II}^{(3)}$ and is given by (2.76).

Case IV. $a_{10} = 0$: In this case there are four branches and we will only give the following example.

$$\begin{aligned}
y_{01} &= -\frac{5}{a_1} : (r_{11}, r_{12}, r_{13}, r_{14}) = (1, 2, 3, 5), \\
y_{02} &= -\frac{10}{a_1} : (r_{21}, r_{22}, r_{23}, r_{24}) = (-2, 1, 2, 5), \\
y_{03} &= -\frac{15}{a_1} : (r_{31}, r_{32}, r_{33}, r_{34}) = (-3, -2, 1, 5), \\
y_{04} &= -\frac{20}{a_1} : (r_{41}, r_{42}, r_{43}, r_{44}) = (-2, -3, -4, 5), \\
y^{(5)} &= -5 \left(yy^{(4)} + 3y'y''' + 2y''^2 + 2y^2y''' + 10yy'y'' + 3y'^3 + 2y^3y'' + 6y^2y'^2 + y^4y' \right) \\
&\quad + A_1 \left(y^{(4)} + 5yy''' + 10y'y'' + 10y^2y' + 15yy'^2 + 10y^3y' + y^5 \right) \\
&\quad + A_{13}(y'' + 2yy') + A_{16}y' + (A'_{13} - A_{13}A_1)y^2 \\
&\quad + (A'_{16} - A''_{13} + A'_1A_{13} + 2A_1A'_{13} - A_1A_{16} - A_1^2A_{13}) + A_{19}, \tag{4.17}
\end{aligned}$$

where A_1 , A_{13} , A_{16} and A_{19} are arbitrary analytic functions of z . Integration of (4.17) once gives the special case of (3.61).

In the procedure used to obtain higher order equations of Painlevé type, the existence of at least one principal branch has been imposed. But, the compatibility conditions at the positive resonances for the secondary branches are identically satisfied for each cases. Instead of having positive distinct integer resonances, one can consider the case of distinct integer resonances. In this case it is possible to obtain equations like Chazy's equation (1.2) which has three negative distinct integer resonances. Starting from the first and second Painlevé equations and by using this procedure one can look for polynomial type equations of any order of Painlevé type having at least one principal branch.

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